SOLUTIONS TO IIT–JEE 2005 (MAINS)
MEMORY BASED QUESTIONS
MATHEMATICS

1. A cricketer plays n matches ($n \geq 1$), total number of runs scored by him is $\left(\frac{n+1}{4}\right)\left(2^{n+1}-n-2\right)$. If he scores $k.2^{n-k+1}$ runs in kth match ($1 \leq k \leq n$). Find the value of n.

[2]

Sol. Total number of runs scored

$$= \sum_{k=1}^{n} k.2^{n-k+1}$$

$$= 2^{n+1}\sum_{k=1}^{n}\frac{k}{2^k}$$

$$= 2^{n+1}\left[1\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+.......+n\left(\frac{1}{2^n}\right)\right]$$

$$= 2\left[2^{n+1}-n-2\right] \qquad \text{(sum of A.G.P)}$$

According to question

$$\left(\frac{n+1}{4}\right)\left(2^{n+1}-n-2\right)=2\left[2^{n+1}-n-2\right]$$

$\Rightarrow$ $\left(\frac{n+1}{4}\right)=2$

$\Rightarrow$ $n = 7$

2. Evaluate $\int_0^{\pi} e^{|\cos x|}\left\{2\sin\left(\frac{\cos x}{2}\right)+3\cos\left(\frac{\cos x}{2}\right)\right\}\sin x\ dx$.

[2]

Sol. $I=\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{\cos x}{2}\right)+3\cos\left(\frac{\cos x}{2}\right)\right)\sin x\ dx$

Put $\cos x = t$

$$= \int_{-1}^{1} e^{|t|}\left(2\sin\left(\frac{t}{2}\right)+3\cos\left(\frac{t}{2}\right)\right)dt$$

$$= 3\int_{-1}^{1} e^{|t|}\cos\frac{t}{2}\,dt \qquad \left(\text{as } e^{|t|}\sin\frac{t}{2} \text{ is odd function}\right)$$

$$= 6\int_0^1 e^t \cos\frac{t}{2}\,dt$$

$$= \frac{24}{5}\left[e\left(\cos\frac{1}{2}+\frac{1}{2}\sin\frac{1}{2}\right)-1\right]$$

3. Find equation of plane which contains the line $2x + y + z - 1 = 0 = x + 2y - z - 4$, and its distance from a point $(2, 1, -1)$ is $\frac{1}{\sqrt{6}}$.

[2]

Sol. Equation of required plane is

$$(2x + y + z - 1) + \lambda(x + 2y - z - 4) = 0$$

Perpendicular distance of this plane from $(2, 1, -1)$ is $\frac{1}{\sqrt{6}}$.

$$\therefore \quad \frac{|\,2(2+\lambda)+1(1+2\lambda)-1(1-\lambda)-(1+4\lambda)\,|}{\sqrt{(2+\lambda)^2+(1+2\lambda)^2+(1-\lambda)^2}} = \frac{1}{\sqrt{6}}$$

$\Rightarrow \quad \lambda = -\frac{8}{5}$

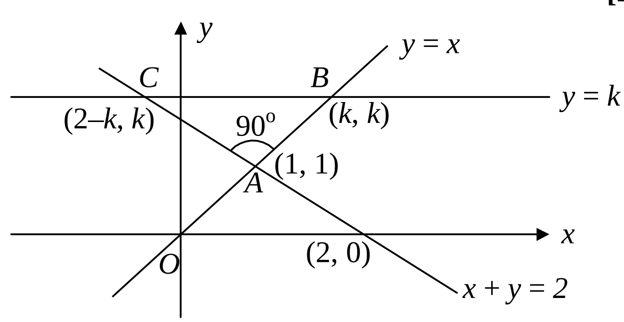
$\therefore$ Equation of plane is

$2x - 11y + 13z - 27 = 0.$

4. A straight line is drawn through point $P(h, k)$ which is parallel to the x–axis and cuts the lines $y = x$ and $x + y = 2$. If area bounded between these three lines is $4h^2$. Find the locus of point P.

[2]

Sol. Required region is ΔABC, which is right angled triangle its area is given by



$$\frac{1}{2}\times\sqrt{2}\,|k-1|\times\sqrt{2}\,|k-1| = 4h^2$$

$$(k-1)^2 = 4h^2$$

Hence locus of point P is

$4x^2 - (y - 1)^2 = 0$

5. A person goes to his office either by car, bus, scooter or train with probability $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}, \frac{1}{7}$ respectively and probability that he reaches his office late is respectively $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}, \frac{1}{9}$. If he reaches his office on time, find the probability that he went by car.

[2]

Sol. Let event $A \equiv$ person goes to office by car

$B \equiv$ person goes to office by bus

$C \equiv$ person goes to office by scooter

$D \equiv$ person goes to office by train

E = reaches office on time

Given $P(A)=\frac{1}{7},\ P(B)=\frac{3}{7}, P(C)=\frac{2}{7}, P(D)=\frac{1}{7}$

$P\left(\frac{E}{A}\right)=\frac{7}{9}, P\left(\frac{E}{B}\right)=\frac{8}{9}, P\left(\frac{E}{C}\right)=\frac{5}{9},\ P\left(\frac{E}{D}\right)=\frac{8}{9}$

Required probability

$$P\left(\frac{A}{E}\right)=\frac{P(A).P\left(\frac{E}{A}\right)}{P(A).P\left(\frac{E}{A}\right)+P(B).P\left(\frac{E}{B}\right)+P(C).P\left(\frac{E}{C}\right)+P(D).P\left(\frac{E}{D}\right)}$$

$$=\frac{\frac{1}{7}\times\frac{7}{9}}{\frac{1}{7}\times\frac{7}{9}+\frac{3}{7}\times\frac{8}{9}+\frac{2}{7}\times\frac{5}{9}+\frac{1}{7}\times\frac{8}{9}}$$

$$=\frac{7}{49}=\frac{1}{7}.$$

6. Three circles of radii 3, 4 and 5 units touches each other externally and the tangents drawn at the point of contact intersect at P. Find distance between point P and point of contact.

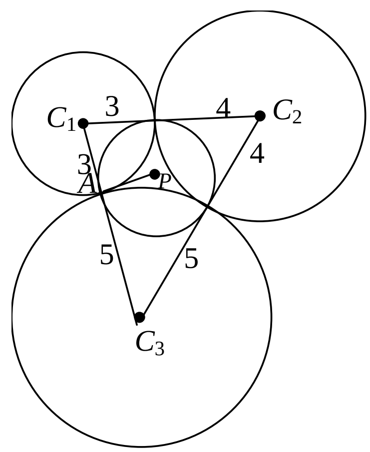
[2]

Sol. Obviously, PA is inradius of triangle $C_1C_2C_3$.

So $r=\frac{\Delta}{s}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

Sides of triangle are 7, 8, 9 and $s=\frac{7+8+9}{2}=12$

So $r=\frac{\sqrt{12.5.4.3}}{12}=\sqrt{5}$

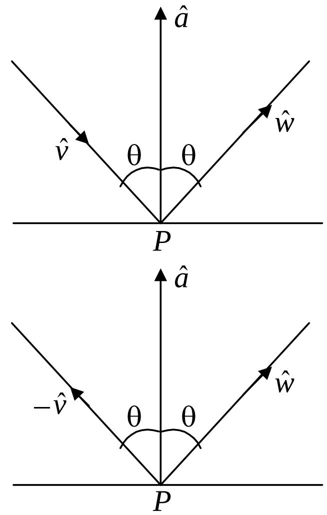


7. If $\hat{v}$ is the unit vector along the incident ray, $\hat{w}$ is the unit vector along the reflected ray and $\hat{a}$ is the unit vector along the outward normal to the plane mirror at the point of incidence P. Find $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.

[2]

Sol. Redrawing these vectors,

Obviously, $\hat{a}$ is the unit vector along internal angular bisector of $-\hat{v}$ and $\hat{w}$



$\therefore \quad \hat{a} = \dfrac{\hat{w} - \hat{v}}{|\hat{w} - \hat{v}|}$

$= \dfrac{\hat{w} - \hat{v}}{2\cos\theta}$

$\therefore \quad \hat{w} = \hat{v} + (2\cos\theta)\hat{a}$

$= \hat{v} - 2(\hat{a}.\hat{v})\hat{a}$

8. If x_1, $x_2 \in R$ and satisfying $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$. Find equation of tangent to this curve at point (1, 2).

[2]

Sol. $\lim\limits_{x_1 \to x_2} \left| \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} \right| \le \lim\limits_{x_1 \to x_2} |x_1 - x_2|$

$\Rightarrow \quad |f'(x_1)| \le 0 \quad \forall\, x_1 \in R$

$\Rightarrow \quad f'(x_1) = 0 \quad \forall\, x_1 \in R$

$\Rightarrow \quad f(x)$ is constant function. Hence equation of tangent at (1, 2) is $y = 2$.

9. From a point, common tangents are drawn to the curve $x^2 + y^2 = 16$ and $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$. Find the slope of common tangent in 1st quadrant and also find the length of intercept between coordinate axes.

[4]

Sol. Any tangent on ellipse is

$y = mx \pm \sqrt{25m^2 + 4}$

If this is also tangent on the circle,

$\left| \dfrac{0 - m \times 0 \pm \sqrt{25m^2 + 4}}{\sqrt{1 + m^2}} \right| = 4$

$\Rightarrow \quad m = \pm \dfrac{2}{\sqrt{3}}$

Since common tangent is in 1st quadrant,

$m = -\dfrac{2}{\sqrt{3}}$

So common tangent in 1st quadrant is

$\sqrt{3}y + 2x = 4\sqrt{7} \qquad (1)$

Intersection of this tangent with x–axis and y–axis are $\left(2\sqrt{7},\, 0\right)$ and $\left(0,\, \frac{4\sqrt{7}}{\sqrt{3}}\right)$

$\therefore$ Length of intercept

$$= \sqrt{\left(2\sqrt{7}-0\right)^2 + \left(0-\frac{4\sqrt{7}}{\sqrt{3}}\right)^2} = \frac{14}{\sqrt{3}}$$

10. From a point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ tangents are drawn to the circle $x^2 + y^2 = 9$. Find the locus of midpoint of chord of contact.

[4]

Sol. Let point on hyperbola be (3secθ, 2tanθ)

Equation of chord of contact is

$$3x\sec\theta + 2y\tan\theta = 9 \qquad (1)$$

Let mid point of chord of contact be (h, k) then

$$hx + ky = h^2 + k^2 \qquad (2)$$

(1) and (2) are identical

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2+k^2}$$

$$\sec\theta = \frac{9h}{3\left(h^2+k^2\right)},\ \tan\theta = \frac{9k}{2\left(h^2+k^2\right)}$$

Now, $\sec^2\theta - \tan^2\theta = 1$

or $$\frac{81}{\left(h^2+k^2\right)^2}\left[\frac{h^2}{9} - \frac{k^2}{4}\right] = 1$$

Hence locus is

$$9\left(4x^2 - 9y^2\right) = 4\left(x^2+y^2\right)^2$$

11. Find all the curves $y = f(x)$, such that length of tangent intercepted between the point of contact and the x–axis is unity.

[4]

Sol. Let point of contact be (x, y)

Equation of tangent at (x, y) is given by

$$Y - y = \frac{dy}{dx}\left(X - x\right)$$

It cuts x–axis at $\left(x - \frac{ydx}{dy},\, 0\right)$

given, $$\sqrt{\left(x - \frac{ydx}{dy} - x\right)^2 + y^2} = 1$$

or $$y^2\left(\left(\frac{dx}{dy}\right)^2 + 1\right) = 1$$

$\Rightarrow \quad \left(\frac{dx}{dy}\right)^2 = \frac{1}{y^2} - 1 = \frac{1-y^2}{y^2}$

$\Rightarrow \quad \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$

$\Rightarrow \quad \frac{\sqrt{1-y^2}}{y}dy = \pm dx$

Integrating both sides we get

$$\sqrt{1-y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1-y^2}-1}{\sqrt{1-y^2}+1}\right) = \pm x + C$$

12. $|z-1| = \sqrt{2}$ is a circle inscribed in a square whose one vertex is $2+i\sqrt{3}$. Find the remaining vertex.

[4]

Sol.

$Z_B = \left(1+i\sqrt{3}\right)e^{i\pi/2} + 1$

$= \left(1+i\sqrt{3}\right)i + 1$

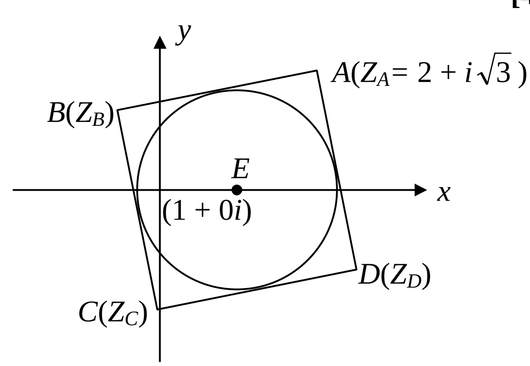
$= \left(1-\sqrt{3}\right) + i$

$Z_C = \left(1+i\sqrt{3}\right)e^{i\pi} + 1$

$= -i\sqrt{3}$

$Z_D = \left(1+i\sqrt{3}\right)e^{-i\pi/2} + 1$

$= \left(1+\sqrt{3}\right) - i$



13. Given $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$. Find real values of $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

[4]

Sol. $x^2(6\sin t - 5) + x(2 - 4\sin t) - (1 + 2\sin t) = 0$

As x is real

$D \ge 0$

$(2-4\sin t)^2 + 4(6\sin t - 5)(1 + 2\sin t) \ge 0$

or $\quad 4\sin^2 t - 2\sin t - 1 \ge 0$

$\Rightarrow$ either $\sin t \le \frac{1-\sqrt{5}}{4}$ or $\sin t \ge \frac{1+\sqrt{5}}{4}$

$\Rightarrow \quad -1 \le \sin t \le \frac{1-\sqrt{5}}{4}$ or $\frac{1+\sqrt{5}}{4} \le \sin t \le 1$

$-\frac{\pi}{2} \le t \le -\frac{\pi}{10}$ or $\frac{3\pi}{10} \le t \le \frac{\pi}{2}$

$\therefore \quad t \in \left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

14. Find the area bounded between the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

[4]

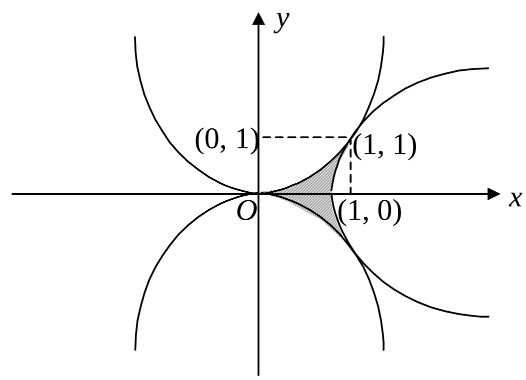
Sol. Solving $y = x^2$ and $y^2 = 4x - 3$,

$x^4 - 4x + 3 = 0$

$\Rightarrow$ $(x-1)^2 (x^2 + 2x + 3) = 0$

$\therefore$ Area = $2\int_0^1 \left(\frac{y^2+3}{4} - \sqrt{y}\right) dy$

$= \frac{1}{3}$ sq. units.



15. If $y = f(x)$ is a cubic polynomial having maxima at $x = -1$ and $f'(x)$ has minima at $x = 1$, also $f(-1) = 10$, $f(1) = -6$. Find the cubic polynomial and also find the distance between the points, which are maxima or minima.

[4]

Sol. Let $f(x) = ax^3 + bx^2 + cx + d$

As $f(-1) = 10$ and $f(1) = -6$ (given)

therefore $-a + b - c + d = 10$ (1)

and $a + b + c + d = -6$ (2)

from (1) and (2)

$\Rightarrow$ $b + d = 2$ (3)

and $a + c = -8$ (4)

Again $f'(x) = 3ax^2 + 2bx + c$

At $x = -1$ $f(x)$ is maximum

$\Rightarrow$ $3a - 2b + c = 0$ (5)

At $x = 1$ $f'(x)$ is minimum

$\therefore$ $f''(1) = 0$

$\Rightarrow$ $6a + 2b = 0$ (6)

Using (3), (4), (5) and (6), we get

$a = 1$, $b = -3$, $c = -9$, $d = 5$

$\therefore$ $f(x) = x^3 - 3x^2 - 9x + 5$

$f'(x) = 3x^2 - 6x - 9$

$= 3(x+1)(x-3)$

Obviously $x = 3$ is a point of minima.

Coordinates of point of maxima and minima are (–1, 10) and (3, –22) respectively.

$\therefore$ Distance between then

$= \sqrt{(3+1)^2 + (-22-10)^2}$

$= \sqrt{1040} = 4\sqrt{65}$.

16. If f and g are two functions such that

$$f(x-y) = f(x)g(y) - f(y).g(x)$$
$$g(x-y) = g(x)g(y) + f(x).f(y)$$

If $\lim_{x \to 0^+} f'(x)$ exists then find $g'(0)$.

[4]

Sol. $f(0) = 0$

$g(0) = 1$

$g(0) = (g(x))^2 + (f(x))^2$

$$g(x) = \sqrt{1-(f(x))^2}$$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \alpha \text{ (exists)} = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

[Because $f(x)$ is odd function. As $f(-y) = -f(y)$]

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-(f(x+h))^2} - \sqrt{1-(f(x))^2}}{h}$$

$$= \lim_{h\to 0} \frac{1-(f(x+h))^2 - 1 + (f(x))^2}{h\left(\sqrt{1-(f(x+h))^2} + \sqrt{1-(f(x))^2}\right)}$$

$$= \lim_{h\to 0} \left\{ \frac{-[f(x+h)-f(x)][f(x+h)+f(x)]}{h\left[\sqrt{1-(f(x+h))^2} + \sqrt{1-(f(x))^2}\right]} \right\}$$

$$\Rightarrow \quad g'(x) = -\alpha \frac{2f(x)}{2\sqrt{1-f(x)^2}}$$

$$\Rightarrow \quad g'(0) = 0$$

17. If $f(x)$ be a differentiable function such that $f'(x) = g(x)$, $g''(x)$ exists, $|f(x)| < 1$ and $(f(0))^2 + (g(0))^2 = 9$. Prove that there is a point 'c' where $c \in (-3, 3)$ such that $g(c).g''(c) < 0$.

[6]

Sol. $f'(x) = g(x)$

$$\int_0^3 g(x)dx = \int_0^3 f'(x)dx = [f(x)]_0^3 = [f(3) - f(0)] \in (-2, 2)$$

$$\int_{-3}^0 g(x)dx = \int_{-3}^0 f'(x)dx = [f(x)]_{-3}^0 = [f(0) - f(-3)] \in (-2, 2)$$

$$(f(0))^2 + (g(0))^2 = 9$$

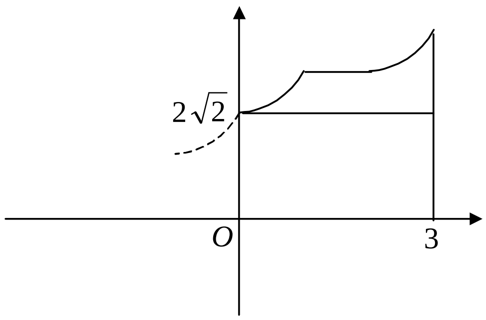
$\Rightarrow \quad |g(0)| > 2\sqrt{2}$ (since $|f(0)| < 1$)

Case I

$g(0) > 2\sqrt{2}$

Let $g''(x) \geq 0$ in $(-3, 3)$

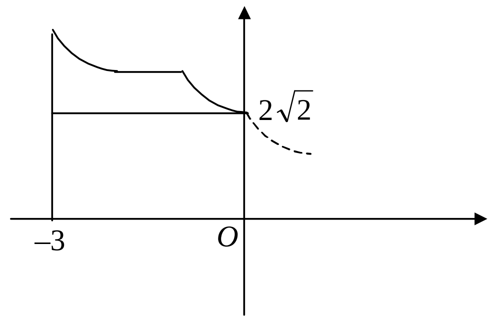
One of the two situations are possible



$\int_0^3 g(x)dx > 6\sqrt{2} > 2$

So contradiction arises

So $g''(x)$ has to be negative somewhere in (0, 3) while $g(x) > 0$ in (0, 3)



$\int_{-3}^0 g(x)dx > 6\sqrt{2} > 2$

So contradiction arises

So $g''(x)$ has to be negative somewhere in (–3, 0) while $g(x) > 0$ in (–3, 0)

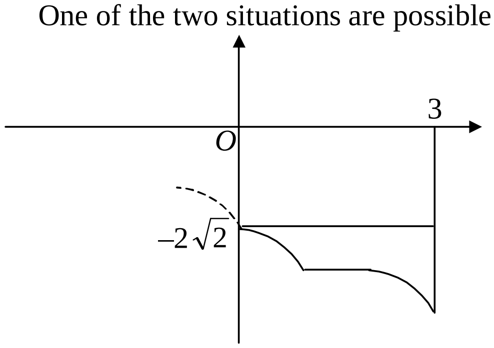
So at least somewhere $g''(x) < 0$ while $g(x) > 0$ in (–3, 3).

Case II

$g(0) < -2\sqrt{2}$

Let $g''(x) \le 0$ in (–3, 3)

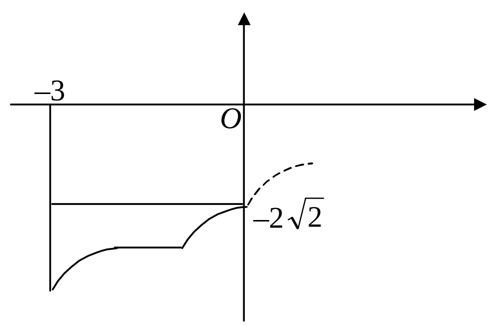
One of the two situations are possible



$\int_0^3 g(x)dx < -6\sqrt{2} < -2$

So contradiction arises

So $g''(x)$ has to be positive somewhere in (0, 3) while $g(x) < 0$ in (0, 3)



$\int_{-3}^0 g(x)dx < -6\sqrt{2} < -2$

So contradiction arises

So $g''(x)$ has to be positive somewhere in (–3, 0) while $g(x) < 0$ in (–3, 0)

S at least somewhere $g''(x) > 0$ while $g(x) < 0$ in (–3, 3).

So at least at one point in (–3, 3),

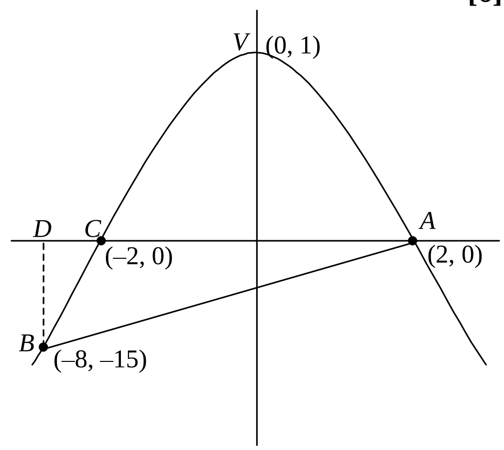
$g(x).g''(x) < 0$

18. If $f(x)$ is a quadratic polynomial and a, b, c are three real and distinct numbers satisfying $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$. Given $f(x)$ cuts the x–axis at A and V is the point of maxima. If AB is any chord which subtends right angle at V, find curve $f(x)$ and area bounded by the chord AB and curve $f(x)$.

[6]

Sol.

$$4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$$
$$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3b$$
$$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$$



Comparing coefficient of a^2, a and constant term on both sides, we get

$f(-1) = \frac{3}{4} = f(1)$ and $f(2) = 0$ (1)

Let $f(x) = Ax^2 + Bx + C$ (2)

from (1) and (2), $A = -\frac{1}{4}$, $B = 0$, $C = 1$.

$\therefore$ $f(x) = -\frac{1}{4}x^2 + 1$

Let $B\left(t, 1 - \frac{t^2}{4}\right)$ be any point on the parabola $f(x) = y = -\frac{x^2}{4} + 1$

As AB chord subtends right angle at V

$\Rightarrow$ $\left(-\frac{1}{2}\right) \times \left(\frac{\frac{t^2}{4}}{-t}\right) = -1$

$\Rightarrow$ $t = -8$

$\therefore$ $B = (-8, -15)$

$\therefore$ Area $(BCVAB)$ = $2 \times \int_0^2 \left(1 - \frac{x^2}{4}\right)dx + \frac{1}{2} \times 10 \times 15 - \left|\int_{-8}^{-2} \left(1 - \frac{x^2}{4}\right)dx\right|$

= $\frac{125}{3}$ sq. units.

Note : *All these questions of IIT–JEE Mains 2005 are based on the memory of the select PIE students who appeared in the examination. PIE Education does not take any responsibility for any sort of discrepancy whatsoever.*